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PROPAGATION OF L_k^q-SMOOTHNESS FOR SOLUTIONS OF THE EULER EQUATION

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ABSTRACT. The motion of an ideal incompressible fluid is described by a system of partial differential equations known as the Euler equation. Considering the initial value problem for this equation, we prove that in a classical solution the L_k^q-regularity of the data propagates along the fluid lines. Our method consists of combining properties of the ε-approximate solution with L^q-energy estimates and simple results of classical singular integral operators. In particular, for the two-dimensional case we present an elementary proof.

1. INTRODUCTION

In [1] Alinhac and Métivier have studied the propagation of analyticity for solutions of the initial value problem for the Euler equation. More precisely, they have shown that the local analyticity of the initial data propagates along the fluid line as long as the existence of a classical solution in L_s^2 is guaranteed (for related results see [2–5, 8]).

On the other hand, Kato and Ponce [10–12] have proved that the Euler equation is well posed (globally when $n = 2$ and locally for $n \geq 3$) in any vector-valued Lebesgue (or Sobolev) spaces $L_s^p = (1-\Delta)^{-s/2}L^p(\mathbf{R}^n)$ with $s > n/p + 1$ and $p \in (1, \infty)$. (The notation $L_s^p(\cdot)$ will be used throughout for vector-valued or scalar functions.)

Related to the above results, in the present paper we are concerned with the propagation of nonsmooth regularities for solutions of the Euler equation. Roughly, we study the problem with data $u_0 \in L_s^p(\mathbf{R}^n)$ such that in an open set $A \subseteq \mathbf{R}^n$ u_0 is more regular, i.e., $u_0|_A \in L_k^q(A)$ with $k - n/q > s - n/p > 1$.

Thus we consider the initial value problem (IVP) for the incompressible Euler equation,

$$(1.1) \qquad \begin{cases} \partial_t u + (u \cdot \nabla)u = -\partial p + f, & x \in \mathbf{R}^n,\ n \geq 2,\ t > 0, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x), \end{cases}$$

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where $\partial_t = \partial/\partial t$, $\nabla = \text{grad} = (\partial/\partial x_1, \dots, \partial/\partial x_n)$, $u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))$ is the vorticity field, $p = p(x,t)$ is the pressure, $(u \cdot \nabla)u$ has j-component $u_k \cdot \partial_k u_j$ (with summation convention), and $f = f(x,t) = (f_1(x,t), \dots, f_n(x,t))$ is a given external force.

For simplicity of the exposition we assume $f \equiv 0$. However, we remark that under an appropriate hypothesis on f the general case can be handled similarly.

In this equation the pressure p is determined (up to a function of t) by u; indeed $\partial p = -(1-P)((u\cdot\nabla)u)$, where P is the projection operator onto the subspace of solenoidal vectors. Therefore the equation in (1.1) may be written as

$$\partial_t u + P((u\cdot\nabla)u) = 0 \tag{1.2.a}$$

or

$$\partial_t u_k + (u\cdot\nabla)u_k = \partial_k \Delta^{-1}(\partial_j u_i \cdot \partial_i u_j) \tag{1.2.b}$$

since P is formally given by the formula $(1 - \partial_i \cdot \partial_j \Delta^{-1})$. Thus P introduces a pseudodifferential operator in the equation. In fact, P is a matrix of a singular integral operator of the Calderón-Zygmund type except for some delta function in the diagonal.

It was proved in [10–12] that for data $u_0 \in PL^p_s(\mathbf{R}^n)$, with $s > n/p + 1$ and $p \in (1,\infty)$, the IVP (1.1) has a unique classical solution $u(\cdot)$ such that $u \in C([0,T] : PL^p_s(\mathbf{R}^n))$. Thus the solution $\phi_t(x_0)$ (fluid lines) of the problem

$$\begin{cases} dx(t)/dt = u(x(t),t), & t \in [0,T], \\ x(0) = x_0 \in \mathbf{R}^n \end{cases} \tag{1.3}$$

is well defined. Moreover, $\phi_t : \mathbf{R}^n \to \mathbf{R}^n$ is a family of diffeomorphisms which preserve volumes.

Next we introduce the following notation. For $A \subseteq \mathbf{R}^n$ open set and any $\delta \geq 0$ we define

$$\begin{aligned} &A_\delta = \{x \in A/\text{distance}\,(x, \partial A) > \delta\}, \\ &A^t_\delta(u_0) = \{\phi_t(x)/x \in A_\delta\} \quad \text{with } t \in [0,T], \text{ and} \\ &D_\delta(u_0) = \{(\phi_t(x), t)/x \in A_\delta,\ t \in [0,T]\}. \end{aligned}$$

For $u_0 \in PL^2_s(\mathbf{R}^n)$ with $s > n/2+1$ and $u_0|_A$ analytic, Alinhac and Métivier [1] showed that the solution $u(\cdot)$ is analytic in $D_0(u_0)$. In their proof sharp estimates for the $L^2(A^t_\delta(u_0))$-norm of all derivatives of the solution were deduced. The existence of these derivatives (the C^∞-result in $D_0(u_0)$) follows from the estimate of Bony in [7].

As mentioned above, here we study the propagation of nonsmooth regularity for solutions of the IVP (1.1) from the evolution equation viewpoint. We will assume that our domain A satisfies the minimal smoothness conditions such that the Calderón extension theorem can be applied (see [13, Chapter VI, Theorem 4]). Our main result is given by the following theorem.

Theorem 1. *Under the above hypothesis on* A, *let* $u_0 \in PL_s^p(\mathbf{R}^n)$, $s > n/p + 1$, $1 < p < \infty$. *If* $u_0|_A \in L_k^q(A)$ *with* $k \in Z^+$, $k \geq s$, *and* $q \geq p$ *then the solution* u *of the IVP* (1.1) *defined in* $\mathbf{R}^n \times [0, T]$ *satisfies*

$$u|_{D_0(u_0)} \in \widetilde{C}([0, T] : L_k^q(A^t(u_0))),$$

i.e., for any $\delta > 0$ *and any* $t \in [0, T]$

$$- \ u(t) \in L_k^q(A_\delta^t(u_0)),$$
$$- \ \lim_{h \to 0} \|\partial^\alpha u(\phi_{t+h}(\cdot), t + h) - \partial^\alpha u(\phi_t(\cdot), t)\|_{L^q(A_\delta)} = 0,$$

for $|\alpha| \leq k$.

Remarks. The L^2-bounded version of Theorem 1 can be proved by combining the results of Bony [7] with Hormander's theorem on propagation of singularities [9]. However, the techniques used here also provide global results (as those in [10–12]) and L^p-estimates (even in a class of unbounded domains)–and in our opinion are simpler.

For the two-dimensional case ($n = 2$) a simple proof of Theorem 1 will be given in §4. In this case T can be taken arbitrarily large.

The conclusion of Theorem 1 can be written in the following equivalent form: for any $\delta > 0$ and any $t \in [0, T]$ there exists $h = h(t, \delta) > 0$ such that u restricted to $[t - h^-, t + h^+] \times A_\delta^t(u_0)$ belongs to $C([t - h^-, t + h^+] \colon L_k^q(A_\delta^t(u_0)))$ where $h^- = \min\{t; h\}$, and $h^+ = \min\{T - t; h\}$.

As in [1], the main difficulties are due to the nonlocal character of the operator P introduced by the pressure p. However, the kernel associated with P is of class C^∞ of the diagonal. Thus one may expect to control the influence of P outside D_0 when estimated in D_0 with a lower norm (with problems on the boundary). Our method of proof combines the results in [11–12] (the continuous dependence of the solution with respect to the initial data, the rate of convergence of the ε-approximation defined below, etc.) With those in [1] (especially Lemma 3.3) and a bootstrap argument which allows us to consider only linear estimates.

The plan of this paper is as follows. In §2 we prepare the results used in the proof of Theorem 1. This will be provided in §3. In §4 we give a different and direct proof of this result for the two-dimensional case. In the appendix we prove some of the results used in the proofs.

2. Preliminary results

Fixing $j \in Z^+$, let $\rho = \rho^j$ be a C^∞ function such that

(a) the support of ρ is contained in the unit ball of $\mathbf{R}^n$,

(b) $\int \rho(x)\,dx = 1$,

(c) $\int x^\alpha \cdot \rho(x)\,dx = 0$ for any multi-indices α with $0 < |\alpha| \leq j$ (the existence of the function $\rho(\cdot)$ will be proved in the appendix).

Define $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ for any $\varepsilon > 0$, and the smoothing operators $(S_\varepsilon)_{\varepsilon>0}$ as $S_\varepsilon g = \rho_\varepsilon * g = g_\varepsilon$ for any $g \in L^1_{\text{loc}}(\mathbf{R}^n)$.

Proposition 2.1. *If* $g \in L^r_l(\mathbf{R}^n)$ *with* $l > 0$ *and* $1 < r < \infty$, *then for any* $\varepsilon > 0$, $g_\varepsilon \in L^r_\infty(\mathbf{R}^n)$ *and*

$$\|g_\varepsilon\|_{l+l',r} \le c \cdot \varepsilon^{-l'} \cdot \|g\|_{l,r} \quad \textit{for any } l' \ge 0, \tag{2.1}$$

$$\|g - g_\varepsilon\|_{l-l',r} \le c \cdot \varepsilon^{l'} \cdot \|g\|_{l,r} \quad \textit{for any } l' \in [0, \min(j, l)]. \tag{2.2.a}$$

Moreover, for a fixed g *the limit when* ε *tends to zero satisfies*

$$\|g - g_\varepsilon\|_{l-l',r} = o(\varepsilon^{l'}) \quad \textit{for any } l' \in [0, \min(j, l)]. \tag{2.2.b}$$

Proof. See [6, 11].

We remark that in (2.2.a) and (2.2.b) it is necessary to consider the minimum between j and l since the moments of the function ρ vanish only up to order j.

The following theorem is a summary of some results proved in [10–12].

Theorem 2.2. *Given* $u_0 \in PL^r_l(\mathbf{R}^n)$ *with* $l > n/r + 1$ *and* $1 < r < \infty$ *there exists* $T > 0$ *such that for any* $\varepsilon > 0$ *the IVP* (1.1) *with initial data* $u_0^\varepsilon = \rho_\varepsilon * u_0$ *has a unique solution* $u^\varepsilon \in C([0, T] : PL^r_l(\mathbf{R}^n))$. *Moreover,*

$$(u^\varepsilon)_{\varepsilon>0} \subseteq C((0, T] : PL^r_\infty(\mathbf{R}^n)), \tag{2.3}$$

$$\operatorname*{Sup}_{[0,T]} \|u^\varepsilon(t)\|_{l,r} \le c \cdot \|u_0^\varepsilon\|_{l,r} \le c \cdot \|u_0\|_{l,r}, \tag{2.4}$$

$$\lim_{\varepsilon \to 0} \operatorname*{Sup}_{[0,T]} \|(u^\varepsilon - u)(t)\|_{l,r} = 0, \tag{2.5}$$

where $u(t)$ *is the solution of* (1.1) *with data* u_0, *and*

$$\operatorname*{Sup}_{[0,T]} \|(u^\varepsilon - u^{\varepsilon'})(t)\|_{l-1,r} \le c \cdot \|u_0^\varepsilon - u_0^{\varepsilon'}\|_{l-1,r} = o(\varepsilon) \tag{2.6}$$

if $j \ge 1$ *(see Proposition* 2.1) *when* $\varepsilon > \varepsilon' > 0$ *tends to zero.*

From Theorem 2.2 and the hypothesis of Theorem 1 it follows that for any $\varepsilon > 0$ the solution $\phi_t^\varepsilon(x_0)$ of the problem

$$\begin{cases} dx(t)/dt = u^\varepsilon(x(t), t), & t \in [0, T], \\ x(0) = x_0 \end{cases}$$

is a well-defined family of diffeomorphisms $\phi_t^\varepsilon : \mathbf{R}^n \to \mathbf{R}^n$.

Proposition 2.3. *Given any $\delta > 0$ there exists $\tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(\delta, A)$ such that for $\varepsilon \in (0, \tilde{\varepsilon}_0)$*

$$D_{2\delta}(u_0^\varepsilon) \subseteq D_\delta(u_0) \subseteq D_{\delta/2}(u_0^\varepsilon). \tag{2.7}$$

Moreover, for any $x, y \in \mathbf{R}^n$

$$\operatorname{dist}(x, y) \sim \operatorname{dist}(\phi_t^\varepsilon(x), \phi_t^\varepsilon(y)) \tag{2.8}$$

uniformly for $\varepsilon \in [0, 1]$ and $t \in [0, T]$.

Next we consider singular integral operators of the form

$$(Tf)(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} \frac{\Omega(y)}{|y|^n} \cdot f(x - y)\, dy = \text{p.v.}(K * f)(x) \tag{2.9}$$

where

(a) Ω is a homogeneous function of degree zero;
(b) Ω has mean value zero on the sphere, i.e.,

$$\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0;$$

(c) Ω is of class C^∞ off the origin.

Similar to the results in [1, Lemmas 3.3 and 3.4], we have

Lemma 2.4. *If $f \in L_l^r(A) \cap L^r(\mathbf{R}^n)$ then for any $\delta > 0$ and any α with $0 < |\alpha| \leq l$*

$$\|\partial^\alpha Tf\|_{r, A_\delta} \leq c \left\{ \|\partial^\alpha f\|_{r, A_{\delta/2}} + \frac{1}{\delta^{|\alpha|}} \cdot \|f\|_r \right\} \tag{2.10}$$

where c does not depend on f or δ.

Proof. Let θ be a function in $C^\infty(\mathbf{R}^1)$ such that $\theta(t) = 1$ if $|t| < \frac{1}{4}$ and $\theta(t) = 0$ if $|t| > \frac{1}{2}$. For $x \in \mathbf{R}^n$ we define $\theta_\delta(x) = \theta(|x|/\delta)$. Thus

$$Tf = K * f = (\theta_\delta \cdot K) * f + ((1 - \theta_\delta)K) * f = K_1 * f + K_2 * f = T_1 f + T_2 f$$

and

$$\partial^\alpha Tf = K_1 * \partial^\alpha f + (\partial^\alpha K_2) * f.$$

It is easy to verify that

$$\|\partial^\alpha K_2\|_1 \leq c/\delta^{|\alpha|} \quad \text{for any } |\alpha| > 0.$$

Therefore

$$\|T_2 f\|_{r, A_\delta} \leq \frac{c}{\delta^{|\alpha|}} \cdot \|f\|_r.$$

On the other hand, the kernel $K_1 = \theta_\delta \cdot K$ satisfies the classical hypothesis for L^p-continuity (see [13, Chapter 2, Theorem 2]), with norm independent of δ, and support of $K_1 \subseteq \{x \in \mathbf{R}^n / |x| \leq \delta/2\}$. Hence

$$\|T_1(\partial^\alpha f)\|_{r, A_\delta} \leq c \cdot \|\partial^\alpha f\|_{r, A_{\delta/2}},$$

which completes the proof.

3. Proof of Theorem 1

In this section we prove our main result, Theorem 1. As in [11–12] we use the ε-approximate solutions u^{ε}, i.e., solutions of the IVP (1.1) with data $u_0^{\varepsilon} = \rho_{\varepsilon} * u_0$. However, we remark that the smoothing operators $S_{\varepsilon}\cdot = \rho_{\varepsilon} * \cdot$ defined here use a different function $\rho(\cdot) = \rho^{j}(\cdot)$ (see the beginning of §2). In the proof of Theorem 1 we fix $j \geq 1$.

Notation. In this section we will use the following notation. For $m \in \mathbf{Z}^{+}$, $r \in (1,\infty]$, $t \in [0,T]$, and $\delta, \varepsilon > 0$

$$\|f(t)\|_{m,r,\delta,\varepsilon} \equiv \|f(t)\|_{L^r_m(A^t_{\delta}(u_0^{\varepsilon}))} = \sum_{|\alpha|\leq m} \left(\int_{A^t_{\delta}(u_0^{\varepsilon})} |\partial_x^{\alpha} f(x,t)|^r \, dx \right)^{1/r}.$$

Also, all constants which do not depend on $\varepsilon \in [0,\varepsilon_0)$ or γ ($\varepsilon_0 = \varepsilon_0(u_0, A)$) will be simply denoted by c.

By simplicity of the exposition we shall consider the most interesting case: $s \in (n/p+1, 2)$. It will be clear from our proof below how to treat the general case.

Proposition 3.1. *Under the above hypothesis, for any* $\varepsilon > 0$

$$u^{\varepsilon}|_{D_0(u_0^{\varepsilon})} \in \widetilde{C}([0,T] : L^q_{\infty}(A^t(u_0^{\varepsilon}))). \tag{3.1}$$

Moreover, for any $m \in \mathbf{Z}^{+}$, $m \geq 1$, *any* $r \geq q$, *and any* $\delta_{0,\gamma} \in (0,\gamma_0)$ *where* $\gamma_0 = \gamma_0(u_0, A) > 0$

$$\sup_{[0,T]} \sup_{[0,\delta_0]} \delta^{m-1} \cdot \|u^{\varepsilon}(t)\|_{m,r,\delta+\gamma,\varepsilon} \leq C \cdot (\|u_0^{\varepsilon}\|_{m,r,\gamma} + \|u_0\|_{s,p}) \tag{3.2}$$

Proof. Using that $u_0 \in L^p_s(\mathbf{R}^n)$ and $r \geq p$ it follows that $u_0 \in L^r_{s_0}(\mathbf{R}^n)$ with $s_0 = s - n/p + n/r > 1 + n/r$. Thus (3.1) and the estimate (3.2) for $m = 0, 1$ are consequences of Theorem 2.2.

To obtain the estimate (3.2) for $m = 2$ we apply the differential operator Δ to the equation in (1.1) written as in (1.2.b). Then

$$\begin{aligned} \frac{d}{dt}\Delta u^{\varepsilon}(\phi_t^{\varepsilon}(x), t) &= \partial_t \Delta u^{\varepsilon} + (u^{\varepsilon}\cdot\nabla)\Delta u^{\varepsilon} \\ &= (u^{\varepsilon}\cdot\nabla)\Delta u^{\varepsilon} - \Delta((u^{\varepsilon}\cdot\nabla)u^{\varepsilon}) + \partial u^{\varepsilon}\cdot\partial^2 u^{\varepsilon}. \end{aligned}$$

Multiplying the above expression by $|\Delta u^{\varepsilon}(\phi_t^{\varepsilon}(x), t)|^{r-1}$, integrating the result in $A_{\delta+\gamma}(u_0^{\varepsilon})$, and using the change of variable $x \to \phi_t^{\varepsilon}(x)$, it follows that

$$\begin{aligned} &\frac{d}{dt}\|\Delta u^{\varepsilon}(t)\|_{0,r,\delta+\gamma,\varepsilon} \\ &\quad\leq c(\|(u^{\varepsilon}\cdot\nabla)\Delta u^{\varepsilon} - \Delta((u^{\varepsilon}\cdot\nabla)u^{\varepsilon})\|_{0,r,\delta+\gamma,\varepsilon} + \|\partial u^{\varepsilon}\cdot\partial^2 u^{\varepsilon}\|_{0,r,\delta+\gamma,\varepsilon}) \\ &\quad\leq c\cdot\|\nabla u^{\varepsilon}(t)\|_{0,\infty,\delta+\gamma,\varepsilon}\cdot(\|\Delta u^{\varepsilon}(t)\|_{0,r,\delta+\gamma,\varepsilon} + \|T_{ijl}\Delta u^{\varepsilon}(t)\|_{0,r,\delta+\gamma,\varepsilon}); \end{aligned} \tag{3.3}$$

above we have used the notation $T_{ijl}\Delta u^{\varepsilon} = \partial_{ij}^2 u_l^{\varepsilon} = -R_i R_j \Delta u_l^{\varepsilon}$ where R_i denotes the ith Riesz transform. For $i \neq j$, T_{ijl} belongs to the class considered in Lemma 2.4. When $i = j$ we have the same result after subtracting a multiple of the identity (see [13, Chapter 3, Theorem 6]). Hence

$$\|T_{ijl}\Delta u^\varepsilon(t)\|_{0,r,\delta+\gamma,\varepsilon} \le c\cdot(\|\Delta u^\varepsilon(t)\|_{0,r,\delta/2+\gamma,\varepsilon} + \tfrac{1}{\delta}\cdot\|u^\varepsilon(t)\|_{1,r}),$$

and from Theorem 2.2,

$$\|u^\varepsilon(t)\|_{1,r} \le c\|u^\varepsilon(t)\|_{s,p} \le c\|u_0^\varepsilon\|_{s,p} \le c\cdot\|u_0\|_{s,p},$$
$$\|\nabla u^\varepsilon(t)\|_{0,\infty,\delta+\gamma,\varepsilon} \le \|\nabla u^\varepsilon(t)\|_\infty \le c\cdot\|u^\varepsilon(t)\|_{s,p} \le c\cdot\|u_0\|_{s,p}.$$

By inserting the above estimates in (3.3), Gronwall's inequality leads to

$$\begin{aligned}\|\Delta u^\varepsilon(t)\|_{0,r,\delta+\gamma,\varepsilon} \le c\cdot\|\Delta u_0^\varepsilon\|_{0,r,\delta+\gamma,\varepsilon} + c\int_0^t \|\Delta u^\varepsilon(\tau)\|_{0,r,\delta/2+\gamma,\varepsilon}\,d\tau \\ + \frac{c}{\delta}\cdot\|u_0\|_{s,p}.\end{aligned}\tag{3.4}$$

Next we introduce a seminorm similar to those used in [1]. Define for $\delta_0 \in (0,\gamma_0)$

$$\Phi^1_{r,\delta_0,\varepsilon}(t) = \sup_{[0,\delta_0]} \delta\cdot\|\Delta u^\varepsilon(t)\|_{0,r,\delta+\gamma,\varepsilon}.$$

Thus by (3.4)

$$\Phi^1_{r,\delta_0,\varepsilon}(t) \le c\cdot(\Phi^1_{r,\delta_0,\varepsilon}(0) + \|u_0\|_{s,p})$$

for $t\in[0,T]$.

From Theorem 2.2 (case $m = 0, 1$) and Lemma 2.4 we obtain that

$$\sup_{[0,T]}\sup_{[0,\delta_0]} \delta\cdot\|u^\varepsilon(t)\|_{2,r,\delta+\gamma,\varepsilon} \le c\cdot(\Phi^1_{r,\delta_0,\varepsilon}(0) + \|u_0\|_{s,p})\tag{3.5}$$

which completes the proof of the case $m = 2$.

To obtain the estimate (3.2) for $m = 3$ we apply the differential operator $\partial\Delta = \Delta\cdot\sum_{|\alpha|\le 1}\partial^\alpha$ to equation (1.2.b). Thus

$$\begin{aligned}\frac{d}{dt}\partial\Delta u^\varepsilon(\phi_t^\varepsilon(x),t) &= \partial_t\partial\Delta u^\varepsilon + (u^\varepsilon\cdot\nabla)\partial\Delta u^\varepsilon \\ &= (u^\varepsilon\cdot\nabla)\partial\Delta u^\varepsilon - \partial\Delta((u^\varepsilon\cdot\nabla)u^\varepsilon) \\ &\quad + \partial u^\varepsilon\cdot T_{ijl}\partial\Delta u^\varepsilon + \partial^2 u^\varepsilon\cdot\partial^2 u^\varepsilon.\end{aligned}$$

Using the argument given above, we have that

$$\begin{aligned}\frac{d}{dt}\|\Delta u^\varepsilon(t)\|_{1,r,\delta+\gamma,\varepsilon} &\le c\cdot\|\nabla u^\varepsilon(t)\|_{0,\infty,\delta+\gamma,\varepsilon} \\ &\quad\cdot(\|\Delta u^\varepsilon(t)\|_{1,r,\delta+\gamma,\varepsilon} + \|T_{ijl}\partial\Delta u^\varepsilon(t)\|_{0,r,\delta+\gamma,\varepsilon}) + \|\partial^2 u^\varepsilon(t)\|^2_{0,2r,\delta+\gamma,\varepsilon}.\end{aligned}$$

Defining

$$\Phi^2_{r,\delta_0,\varepsilon}(t) = \sup_{[0,\delta_0]} \delta^2\cdot\|\Delta u^\varepsilon(t)\|_{1,r,\delta+\gamma,\varepsilon},$$

by Lemma 2.4 and Gronwall's inequality it follows that

$$\tag{3.6} \begin{aligned} \Phi^2_{r,\delta_0,\varepsilon}(t) \le c\cdot(\Phi^2_{r,\delta_0,\varepsilon}(0) + \|u_0\|_{s,p} + \int_0^t \Phi^2_{r,\delta_0,\varepsilon}(\tau)\,d\tau \\ + \sup_{[0,T]}\sup_{[0,\delta_0]} \delta^2\|\partial^2 u^\varepsilon(t)\|^2_{0,2r,\delta+\gamma,\varepsilon}). \end{aligned}$$

Estimate (3.5) and Theorem 2.2 lead to

$$\begin{aligned} \sup_{[0,T]}\sup_{[0,\delta_0]} \delta^2\cdot\|\partial^2 u^\varepsilon(t)\|^2_{0,2r,\delta+\gamma,\varepsilon} &\le c\cdot \sup_{[0,\delta_0]}\delta^2\cdot\|\partial^2 u_0^\varepsilon\|^2_{0,2r,\delta+\gamma} + c\cdot\|u_0\|_{s.p} \\ &\le c\cdot\sup_{[0,\delta_0]}\delta^2\cdot\|\partial u_0^\varepsilon\|_{2,r,\delta+\gamma} + c\cdot\|u_0\|_{s,p} \\ &\le c\cdot\sup_{[0,\delta_0]}\delta^2\cdot\|\Delta u_0^\varepsilon\|_{1,r,0} + c\cdot\|u_0\|_{s,p} \\ &= c\cdot\Phi^2_{r,\delta_0,\varepsilon}(0) + c\cdot\|u_0\|_{s,p}. \end{aligned}$$

Above we have used Calderón's extension theorem, Gagliardo-Nirenberg's inequality, Lemma 2.4, and the existence of a constant c (which does not depend on $\varepsilon\in[0,\varepsilon_0)$) such that $c\|\Delta u_0^\varepsilon\|_{1,r,\delta_0+\gamma}$ is larger than $\|\Delta u_0\|_{1,r,0}$.

By inserting the previous estimate in (3.6) we obtain that

$$\Phi^2_{r,\delta_0,\varepsilon}(t) \le c\cdot(\Phi^2_{r,\delta_0,\varepsilon}(0) + \|u_0\|_{s,p})$$

which proves (3.2) for the case $m=3$.

The proof for larger m uses the bootstrap argument given above.

Proof of Theorem 1.

Case $k=2$. For $\varepsilon>\varepsilon'>0$, define $y(t) = y^{\varepsilon,\varepsilon'}(t) = (u^{\varepsilon'} - u^\varepsilon)(t)$. Then

$$\partial_t y + (u^{\varepsilon'}\cdot\nabla)y = -(y\cdot\nabla)u^\varepsilon + \partial\Delta^{-1}((\partial u^{\varepsilon'} + \partial u^\varepsilon)\cdot\partial y).$$

By the argument given in the proof of Proposition 3.1 we obtain that

$$\begin{aligned} \frac{d}{dt}\|\Delta y(t)\|_{0,q,\delta+\gamma,\varepsilon'} \le\ & c\cdot(\|\partial u^\varepsilon(t)\|_\infty + \|\partial u^{\varepsilon'}(t)\|_\infty) \\ & \cdot(\|\Delta y(t)\|_{0,q,\delta+\gamma,\varepsilon'} + \|T_{ijl}\Delta y(t)\|_{0,q,\delta+\gamma,\varepsilon'}) \\ & + (\|u^\varepsilon(t)\|_{2,q,\delta+\gamma,\varepsilon'} + \|u^{\varepsilon'}(t)\|_{2,q,\delta+\gamma,\varepsilon'}) \\ & \cdot\|\partial y(t)\|_{0,\infty,\delta+\gamma,\varepsilon'} + \|\nabla u^\varepsilon(t)\|_{2,q,\delta+\gamma,\varepsilon'}\cdot\|y(t)\|_{0,\infty,\delta+\gamma,\varepsilon'}. \end{aligned}$$

By Propositions 3.1 and 2.3

$$\|u^{\varepsilon'}(t)\|_{2,q,\delta+\gamma,\varepsilon'} \le c\cdot\delta^{-1},$$

and for $\varepsilon\in(0,\tilde\varepsilon_0)$ where $\tilde\varepsilon_0 = \tilde\varepsilon_0(\gamma,A)$

$$\|u^\varepsilon(t)\|_{2,q,\delta+\gamma,\varepsilon'} \le c\cdot\|u^\varepsilon(t)\|_{2,q,\delta+\gamma/2,\varepsilon} \le c\cdot\delta^{-1},$$

$$\|\partial u^\varepsilon(t)\|_{2,q,\delta+\gamma,\varepsilon'} \le c\cdot\|\partial u^\varepsilon(t)\|_{2,q,\delta+\gamma/2,\varepsilon} \le c\cdot\delta^{-2}\cdot\varepsilon^{-1}.$$

Also by Theorem 2.2

$$\begin{aligned}
&\|\partial u^{\varepsilon}(t)\|_{\infty} + \|\partial u^{\varepsilon'}(t)\|_{\infty} \le c \cdot \|u_0\|_{s,p}, \\
&\|y(t)\|_{\infty} \le c \cdot \|y(t)\|_{s-1,p} \le c \cdot \|u_0^{\varepsilon} - u_0^{\varepsilon'}\|_{s-1,p} = o(\varepsilon), \\
&\|\partial y(t)\|_{\infty} \le c \cdot \|y(t)\|_{s,p} = o(1)
\end{aligned}$$

when ε tends to zero.

Thus introducing the seminorm

$$\psi_{\delta_0}(t') = \sup_{[0,t']}\sup_{[0,\delta_0]} \delta^2 \cdot \|\Delta y(t)\|_{0,q,\delta+\gamma,\varepsilon'},$$

with the argument given in the previous proof we find that for any $\gamma \in (0, \gamma_0)$ fixed

$$\psi_{\delta_0}(T) \le c(\psi_{\delta_0}(0) + o(1)) = o(1)$$

when ε tends to zero. Again using Lemma 2.4 and Theorem 2.2, it follows that for any $\beta > 0$

$$\lim_{\varepsilon,\varepsilon' \to 0} \sup_{[0,T]} \|(u^{\varepsilon} - u^{\varepsilon'})(t)\|_{2,q,\beta,\varepsilon'} = 0.$$

In particular, we have that for any $\beta > 0$ and any $t \in [0, T]$, (u^{ε}) restricted to $[t - h^-, t + h^+] \times A_{2\beta}^t(u_0)$ is a Cauchy sequence in $C([t - h^-, t + h^+] : L_2^q(A_{2\beta}^t(u_0)))$ where $h = h(\beta) > 0$. Since β is arbitrary and (u^{ε}) converges pointwise to u (Theorem 2.2), the proof of Theorem 1 when $k = 2$ is completed.

Case $k = 3$. In this case by the Calderón extension theorem, the Sobolev imbedding theorem, and the result for $k = 2$ it follows that

$$\sup_{[0,T]}\sup_{[0,\delta_0]} \delta^2 \cdot \|(u^{\varepsilon} - u^{\varepsilon'})(t)\|_{2,2q,\delta+\gamma,\varepsilon'} = o(1),$$

and by (3.2)

$$\sup_{[0,T]}\sup_{[0,\delta_0]} \delta^2 \|\partial^2 u^{\varepsilon}(t)\|_{0,2q,\delta+\gamma,\varepsilon'} \le c.$$

Combining these estimates with the techniques given in the proof of the previous case we obtain the desired result.

The proof for larger k follows by reapplying the bootstrap argument given above.

4. A different proof of Theorem 1 when $n = 2$

Using the structure of the vorticity equation we give a simpler proof of Theorem 1 for the two-dimensional case. Consider the IVP for the linear scalar equation

$$\begin{cases} \partial_t v + (a \cdot \nabla) v = 0, & x \in \mathbf{R}^2,\ t \in [0, T], \\ v(x, 0) = v_0(x), \end{cases} \tag{4.1}$$

with given coefficient $a \in C([0, T] : L_l^r(\mathbf{R}^2; \mathbf{R}^2))$, $l > 2/r + 1$, and $r \in (1, \infty)$.

Lemma 4.1. *For any* $v_0 \in L^r_{l'}(\mathbf{R}^2 : \mathbf{R})$ *with* $l' \geq 1$ *the IVP* (4.1) *has a unique solution* $v(x,t)$. *Moreover,* $v \in C([0,T] : L^r_{l'_0}(\mathbf{R}^2 : \mathbf{R}))$ *where* $l'_0 = \min\{l', [l]\}$.

Proof. See [10].

In our case we start with $a = u \in C([0,T] : L^q_{s_0})$ (Theorem 2.2), and $v_0 = (\nabla x u_0) \cdot \chi_{A_\delta} = \omega_0 \cdot \chi_{A_\delta} \in L^q_{k-1}(\mathbf{R}^2)$ where $\chi_{A_\delta} \in C^\infty(\mathbf{R}^2)$ with $\chi_{A_\delta}(x) = 1$ if $x \in A_\delta$ and $\chi_{A_\delta}(x) = 0$ if $x \notin A$, for $\delta > 0$ arbitrary.

Using Lemma 4.1 the solution $\tilde{v} \in C([0,T] : L^q_\theta)$ with $\theta = \min\{[s_0], k-1\}$. It is easy to check that $\tilde{v}(x,t) = \nabla x u(x,t) = \omega(x,t)$ for $(x,t) \in D_\delta$. Thus, by the Biot-Savart law $(u = -\nabla \times \Delta^{-1}\omega)$ and Lemma 2.4 it follows that

$$u|_{D_0(u_0)} \in \widetilde{C}([0,T] : L^q_{\theta+1}(A^t(u_0))).$$

If $\theta = k-1$, we have finished the proof. Otherwise we reapply the argument with $v_0 = \omega_0 \cdot \chi_{A_\delta}$ and $a = a(x,t) = u(x,t) \cdot b_\delta(x,t)$ where $b_\delta(x,t) \in C^\infty(\mathbf{R}^2 \times [0,T])$ satisfies

$$b_\delta(\cdot,t) = \begin{cases} 1, & x \in A^t_\delta, \\ 0, & x \notin A^t_{\delta/2}; \end{cases}$$

therefore $a \in C([0,T] : L^q_{[s_0]+1})$.

After using this argument $k - [s_0]$ times we finish the proof.

Appendix

We will show that given any $j \in \mathbf{Z}^+$ there exists $\rho^j = \rho \in C^\infty(\mathbf{R}^n)$ such that

(a) ρ is supported in the unit ball,

(b) $\int \rho(x)\,dx \neq 0$,

(c) $\int x^\alpha \cdot \rho(x)\,dx = 0$ for any $\alpha \in (\mathbf{Z}^+)^n$ such that $0 < |\alpha| \leq j$.

Let E be the space of the functions in $C^\infty(\mathbf{R}^1)$ with support in $(0,1)$. Define $L: E \to \mathbf{R}^{n+j+1}$ as

$$L(f) = \left(\int f(x)\,dx\,;\ \int x \cdot f(x)\,dx\,;\ \cdots\,;\ \int x^{n+j} \cdot f(x)\,dx\right).$$

Clearly L is linear and its kernel is nontrivial. Let f_0 be a function in $\operatorname{Ker}(L) - \{0\}$. By the Paley-Wiener theorem there exists $k \in \mathbf{Z}^+$ $(k > n+j)$ such that $\int x^k \cdot f_0(x)\,dx \neq 0$. Let k_0 be the smallest among all possible k's. Defining $f_1(x) = x^{k_0-(n+j)} \cdot f_0(x); \cdots ; f_{n+j+1}(x) = x^{k_0} \cdot f_0(x) \in E$, we find that L is onto. Therefore there exists $\tilde{f} \in E$ such that $L\tilde{f} = (0, \ldots, 1, \ldots, 0) = e_{n-1}$.

Taking $\rho(x) = \tilde{f}(|x|)$ we obtain the desired function.

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